MATH 2050C Lecture on 2/19/2020

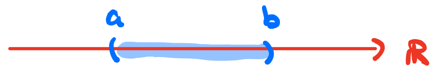
Announcement: Problem Set 3 due this Friday (Feb 21).

Recall: $\mathbb{R}$ is a complete ordered field. (Existence? See Rudin)

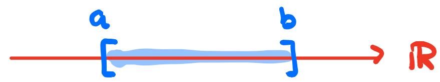
Intervals (§ 2.5)

- 4 kinds of bdd intervals (Let $a, b \in \mathbb{R}$, $a < b$)

open $(a,b) := \{x \in \mathbb{R} \mid a < x < b\}$



closed $[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$

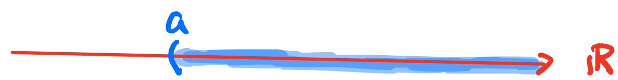


half open & closed
$(a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}$
$[a,b) := \{x \in \mathbb{R} \mid a \leq x < b\}$

Define: $\text{Length}(I) := b - a \quad (< +\infty)$

- 5 kinds of unbdd intervals

$(a, +\infty) := \{x \in \mathbb{R} \mid a < x\}$

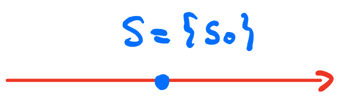


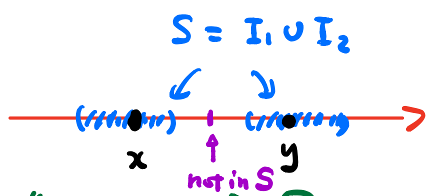
Similarly, $[a, +\infty)$, $(-\infty, b)$, $(-\infty, b]$, $(-\infty, +\infty) = \mathbb{R}$

Note: $\pm\infty \notin \mathbb{R}$ but just a symbol.

Q: When is $S \subseteq \mathbb{R}$ an interval?

Non-examples:





Idea: "intervals" $\approx$ "connected" pieces of $\mathbb{R}$

Thm: (Characterization of intervals)

Let $S \subseteq \mathbb{R}$. Suppose (i) $\exists\, s_1, s_2 \in S$, $s_1 \neq s_2$ (say $s_1 < s_2$)

(ii) If $x, y \in S$, $x < y$, then $[x,y] \subseteq S$.

Then, S is an interval. [**Note:** could be open / closed, bdd / unbdd]

Proof: We just treat the case that S is bdd. (Ex: unbdd. case)

- Completeness of $\mathbb{R}$ $\Rightarrow$ $a := \inf S$, $b := \sup S$ exist in $\mathbb{R}$.
- (i) $\Rightarrow$ $a \le s_1 < s_2 \le b$ $\Rightarrow$ $a < b$.
- Claim: $(a,b) \subseteq S$ (i.e. $\forall x \in (a,b)$, we have $x \in S$)

  Pf of Claim: Let $x \in (a,b)$.

  - $x > a$ $\Rightarrow$ x is NOT a lower bd

    $\Rightarrow$ $\exists s' \in S$ s.t. $s' < x$
  - $x < b$ $\Rightarrow$ x is NOT an upper bd

    $\Rightarrow$ $\exists s'' \in S$ s.t. $x < s''$

  So, $x \in (s', s'') \subseteq S$ by (ii). In particular, $x \in S$.
- Therefore, we conclude that S is one of the following:

$$(a,b) \quad \text{or} \quad [a,b) \quad \text{or} \quad (a,b] \quad \text{or} \quad [a,b]$$

$\square$

Thm: (Nested Interval Property)

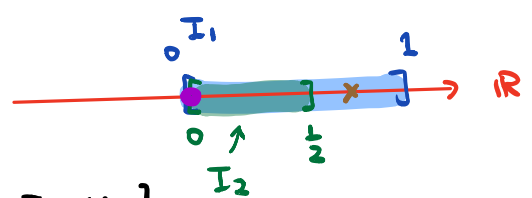
Let $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, be a "sequence" of closed & bdd intervals and they are "nested":

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots\cdots$$

Then, $\displaystyle\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

Furthermore, if $\inf_n \text{Length}(I_n) = 0$, then $\displaystyle\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Picture: E.g.) $I_n := [0, \frac{1}{n}]$



$$\{0\} = \bigcap_{n=1}^{\infty} I_n := \{x \in \mathbb{R} \mid x \in I_n \ \forall n\}$$

E.g.) $I_n := [0, 1+\frac{1}{n}]$. $\bigcap_{n=1}^{\infty} I_n = [0,1] \neq \phi$

Non-E.g.) $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$ open interval

Non-E.g.) $\bigcap_{n=1}^{\infty} [n, +\infty) = \phi$

unbdd